THE SUPER FIXED POINT PROPERTY FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

ANDRZEJ WIŚNICKI

Abstract. We show that the super fixed point property for nonexpansive mappings and for asymptotically nonexpansive mappings in the intermediate sense are equivalent. As a consequence, we obtain fixed point theorems for asymptotically nonexpansive mappings in uniformly nonsquare and uniformly noncreasy Banach spaces. The results are generalized for commuting families of asymptotically nonexpansive mappings.

1. Introduction

The classical problem in metric fixed point theory, a branch of fixed point theory which emerged from the Banach contraction principle, is concerned with the existence of fixed points of nonexpansive mappings. Recall that a mapping $T : C \to C$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. A Banach space X is said to have the fixed point property (FPP for short) if every nonexpansive self-mapping defined on a nonempty bounded closed and convex set $C \subset X$ has a fixed point (see [2, 10, 11]). One of the natural and extensively studied generalizations of nonexpansive mappings was introduced by Goebel and Kirk [9]. A mapping $T : C \to C$ is said to be asymptotically nonexpansive if there exists a sequence of real numbers (k_n) with $\lim_n k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Let B be the closed unit ball in ℓ_2 and set

$$T(x_1, x_2, x_3, ...) = (0, x_1^2, a_2 x_2, a_3 x_3, ...),$$

where $(x_1, x_2, x_3, ...) \in B$ and (a_n) is a sequence of reals in $(0, 1)$ such that $\prod_{n=2}^{\infty} a_n = \frac{1}{2}$. Then

$$\|Tx - Ty\| \leq 2\|x - y\|$$

and

$$\|T^n x - T^n y\| \leq 2 \prod_{i=2}^{n} a_i \|x - y\|$$

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(see [9]). This shows that the class of asymptotically nonexpansive mappings is wider than the class of nonexpansive mappings.

In spite of the common belief that asymptotically nonexpansive mappings share a lot of properties of nonexpansive mappings, there exist relatively few results concerning the existence of fixed points for such mappings. The original result from [9] stating that asymptotically nonexpansive mappings have the fixed point property in uniformly convex spaces was generalized in [22] when X is nearly uniformly convex, in [17] when X satisfies the uniform Opial condition and in [15] when X has uniform normal structure. It is still unknown whether normal structure implies the fixed point property for asymptotically nonexpansive mappings acting on a convex and weakly compact subset of a Banach space X. Until now, the situation has been even worse in Banach spaces without normal structure.

In 1998, Kirk, Martinez Yañez and Shin [16] showed that if X has the super fixed point property for nonexpansive mappings (i.e., every Banach space finitely representable in X has FPP), then every asymptotically nonexpansive mapping defined on a bounded closed and convex subset of X has approximate fixed points, i.e., there exists a sequence (x_n) such that $\lim_n \|Tx_n - x_n\| = 0$. In the present paper we strengthen this result by showing, in Theorem 2.4, that the super fixed point property for nonexpansive mappings is equivalent to the super fixed point property for asymptotically nonexpansive mappings in the intermediate sense (see Section 2 for the definition). In particular, we obtain fixed point theorems for asymptotically nonexpansive mappings in both uniformly nonsquare and uniformly noncreasy Banach spaces. In Section 3, the above results are extended for commuting families of asymptotically nonexpansive mappings in the intermediate sense.

It was shown in [6, Th. 10] that every Banach space X which contains an isomorphic copy of c_0 fails the fixed point property for asymptotically nonexpansive mappings. Our results support the conjecture that the fixed point property for nonexpansive mappings and for asymptotically nonexpansive mappings are equivalent which would imply the failure of the FPP inside isomorphic copies of c_0.

2. Main result

Let X and Y be Banach spaces and let $0 < \varepsilon < 1$. A linear map $T : Y \to X$ is an ε-isometry if

$$(1-\varepsilon)\,\|y\| \le \|Ty\| \le (1+\varepsilon)\,\|y\|$$

for all $y \in Y$. Recall that Y is said to be finitely representable in X if for each $\varepsilon \in (0,1)$ and every finite dimensional subspace $M \subset Y$ there exists an ε-isometry $T : M \to X$.

We say that X is superreflexive if every Banach space Y which is finitely representable in X is reflexive. A Banach space X has the super fixed point property for nonexpansive mappings (SFPP) if every Banach space Y which is finitely representable in X has FPP. It follows from the result of van Dulst and Pach [7, Th. 3.2] that SFPP implies superreflexivity.

The notion of finite representability is closely related with the construction of the Banach space ultrapower. Let $\mathcal{U}$ be an ultrafilter defined on a set I. The ultrapower $\widetilde{X}$ (or $(X)_{\mathcal{U}}$) of a Banach space X is the quotient space of

$$l_\infty(X) = \left\{ (x_n) : x_n \in X \text{ for all } n \in I \text{ and } \|(x_n)\| = \sup_n \|x_n\| < \infty \right\}$$

by

$$\left\{ (x_n) \in l_\infty(X) : \lim_{n\to\mathcal{U}} \|x_n\| = 0 \right\}.$$

Here $\lim_{n\to\mathcal{U}}$ denotes the ultralimit over $\mathcal{U}$. One can prove that the quotient norm on $(X)_{\mathcal{U}}$ is given by

$$\|(x_n)_{\mathcal{U}}\| = \lim_{n\to\mathcal{U}} \|x_n\|,$$

where $(x_n)_{\mathcal{U}}$ is the equivalence class of (x_n). It is also clear that X is isometric to a subspace of $(X)_{\mathcal{U}}$ by the mapping $x \to (x)_{\mathcal{U}}$.

The connection between ultrapowers and finite representability was observed independently by Henson and Moore [13], and Stern [21] (see also [1, 12, 20]).

Theorem 2.1. *A Banach space Y is finitely representable in X if and only if there exists an ultrafilter $\mathcal{U}$ such that Y is isometric to a subspace of $(X)_{\mathcal{U}}$.*

It follows from the above theorem that X has SFPP iff every ultrapower $(X)_{\mathcal{U}}$ has FPP.

In 1980, the Banach space ultrapower construction was applied in fixed point theory by Maurey [18] who proved the fixed point property for all reflexive subspaces of $L_1[0,1]$ and the weak fixed point property for c_0 and H^1. Inspired by [16], we apply this construction to asymptotically nonexpansive mappings in a slightly more general setting. Recall that a mapping $T : C \to C$ is said to be asymptotically nonexpansive in the intermediate sense if T is continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\|T^n x - T^n y\| - \|x - y\|\right) \le 0 \tag{1}$$

(in the original definition, in [4], T was assumed to be uniformly continuous). In particular, the condition (1) is satisfied if $\limsup_{n\to\infty} |T^n| \le 1$, where $|T^n|$ denotes the (exact) Lipschitz constant of T^n (and C is bounded).

Let C be a nonempty bounded closed and convex subset of a Banach space X and $T : C \to C$ be asymptotically nonexpansive in the intermediate sense. Take a free ultrafilter p on $\mathbb{N}$ and denote by $\widetilde{C} \subset (X)_p$ the set

$$\widetilde{C} = \{(x_n)_p \in (X)_p : x_n \in C \text{ for all } n \in \mathbb{N}\}.$$

Let $(\mathcal{N}, \preceq)$ be a directed set, where

$$\mathcal{N} = \left\{ (\alpha_n) \in \mathbb{N}^{\mathbb{N}} : \alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots \right\}$$

is a family of all increasing sequences of natural numbers directed by the relation $(\alpha_n) \preceq (\beta_n)$ iff $\alpha_n \le \beta_n$ for every $n \in \mathbb{N}$. Notice that if $(x_n)_p, (y_n)_p \in$

$\widetilde{C}$ and $(\alpha_n) \in \mathcal{N}$, then

$$\lim_{n\to p}(\|T^{\alpha_n}x_n - T^{\alpha_n}y_n\| - \|x_n - y_n\|) \le \limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0.$$

Therefore, we may extend the mapping T by setting, unambiguosly,

$$\widehat{T}_{(\alpha_n)}(x_n)_p = (T^{\alpha_n}x_n)_p. \tag{2}$$

It is not difficult to see that $\widehat{T}_{(\alpha_n)} : \widetilde{C} \to \widetilde{C}$ is nonexpansive for every $(\alpha_n) \in \mathcal{N}$. For $x \in C$, we shall write $\dot{x} = (x)_p = (x, x, ...)_p$.

Lemma 2.2. *Let $T : C \to C$ be asymptotically nonexpansive in the intermediate sense and suppose that there exists $\widetilde{y} \in \widetilde{C}$ such that*

$$\widehat{T}_{(\alpha_n)}\widetilde{y} = \widetilde{y} \tag{3}$$

for all $(\alpha_n) \in \mathcal{N}$. Let $\|\widetilde{y} - \dot{x}_0\| < \delta$ for some $x_0 \in C$ and $\delta > 0$. Then, for every $\varepsilon > 0$ there exist $x \in C$ and $n_0 \in \mathbb{N}$ such that $\|x - x_0\| < \delta$ and $\|T^n x - x\| < \varepsilon$ for every $n \ge n_0$.

Proof. Since $\|\widetilde{y} - \dot{x}_0\| < \delta$, there exists a sequence (y_n) in C such that $\|y_n - x_0\| < \delta$ for all $n \in \mathbb{N}$ and $\widetilde{y} = (y_n)_p$. Assume, conversely to our claim, that there exists $\varepsilon_0 > 0$ such that for every $x \in C$ and $n_0 \in \mathbb{N}$ there exists $n \ge n_0$ such that $\|x - x_0\| \ge \delta$ or $\|T^n x - x\| \ge \varepsilon_0$. We shall define a sequence (β_n) by induction. For $n = 0$ and $y_0 \in C$, there exists β_0 such that $\|T^{\beta_0}y_0 - y_0\| \ge \varepsilon_0$. Suppose that we have chosen $\beta_0 < \beta_1 < ... < \beta_n$ such that $\|T^{\beta_i}y_i - y_i\| \ge \varepsilon_0$ for $i = 0, 1, ..., n$. By assumption, since $\|y_{n+1} - x_0\| < \delta$, there exists $\beta_{n+1} > \beta_n$ such that $\|T^{\beta_{n+1}}y_{n+1} - y_{n+1}\| \ge \varepsilon_0$. (To be more precise, we can define, for example, β_{n+1} as the minimum of $\{\beta > \beta_n : \|T^\beta y_{n+1} - y_{n+1}\| \ge \varepsilon_0\}$). Thus we obtain a sequence $(\beta_n) \in \mathcal{N}$ such that $\|T^{\beta_n}y_n - y_n\| \ge \varepsilon_0$ for all $n \in \mathbb{N}$. Hence $\left\|\widehat{T}_{(\beta_n)}\widetilde{y} - \widetilde{y}\right\| \ge \varepsilon_0$, a contradiction with (3). □

A Banach space X is said to have the fixed point property for asymptotically nonexpansive mappings (in the intermediate sense) if every asymptotically nonexpansive (in the intermediate sense) self-mapping acting on a nonempty bounded closed and convex set $C \subset X$ has a fixed point.

Theorem 2.3. *Assume that X has the super fixed point property for nonexpansive mappings. Then X has the fixed point property for asymptotically nonexpansive mappings in the intermediate sense.*

Proof. Assume that X has the super fixed point property for nonexpansive mappings. Let $T : C \to C$ be an asymptotically nonexpansive mapping in the intermediate sense acting on a nonempty bounded closed and convex set $C \subset X$. By [7, Th. 3.2], X is superreflexive and hence C is weakly compact. Without loss of generality we can assume that $\operatorname{diam} C = 1$. Take a free ultrafilter p on $\mathbb{N}$, $(\alpha_n) \in \mathcal{N}$, and define $\widehat{T}_{(\alpha_n)}$ by (2). Notice that for every $(\alpha_n), (\beta_n) \in \mathcal{N}$ and any $(z_n)_p \in \widetilde{C}$,

$$(\widehat{T}_{(\alpha_n)} \circ \widehat{T}_{(\beta_n)})(z_n)_p = (T^{\alpha_n}T^{\beta_n}z_n)_p = (\widehat{T}_{(\beta_n)} \circ \widehat{T}_{(\alpha_n)})(z_n)_p.$$

It follows from the Bruck theorem (see [3, Th. 1]) that there exists $\widetilde{y}_0 \in \widetilde{C}$ such that $\widehat{T}_{(\alpha_n)}\widetilde{y}_0 = \widetilde{y}_0$ for all $(\alpha_n) \in \mathcal{N}$ (a similar argument but for two mappings was used in [16, Th. 4.1]). Fix $\varepsilon < 1$ and $x_0 \in C$. We shall define by induction a sequence (n_j) of natural numbers and a sequence (x_j) of elements in C such that

$$\|x_j - x_{j-1}\| < 3\varepsilon^{j-1} \text{ and } \|T^n x_j - x_j\| < \varepsilon^j \text{ for every } n \geq n_j, j \geq 1. \quad (4)$$

By Lemma 2.2, there exist $x_1 \in C$ and $n_1 \in \mathbb{N}$ such that $\|T^n x_1 - x_1\| < \varepsilon$ for every $n \geq n_1$ and $\|x_1 - x_0\| \leq \operatorname{diam} C < 3$.

Suppose that we have chosen natural numbers $n_1, ..., n_j$ and $x_1, ..., x_j \in C$ $(j \geq 1)$ such that

$$\|x_i - x_{i-1}\| < 3\varepsilon^{i-1} \text{ and } \|T^n x_i - x_i\| < \varepsilon^i \text{ for every } n \geq n_i, 1 \leq i \leq j.$$

Let

$$D_j = \left\{ \widetilde{y} = (y_n)_p \in \widetilde{C} : \limsup_{(\alpha_n)\in\mathcal{N}} \|\widehat{T}_{(\alpha_n)}\dot{x}_j - \widetilde{y}\| \leq \varepsilon^j \right\},$$

where $\dot{x}_j = (x_j, x_j, ...)_p$ and

$$\limsup_{(\alpha_n)\in\mathcal{N}} \|\widehat{T}_{(\alpha_n)}\dot{x}_j - \widetilde{y}\| = \inf_{(\alpha_n)\in\mathcal{N}} \sup_{(\beta_n)\succeq(\alpha_n)} \lim_{n\to p} \left\|T^{\beta_n} x_j - y_n\right\|$$

denotes the upper limit of the net $(\|\widehat{T}_{(\alpha_n)}\dot{x}_j - \widetilde{y}\|)_{(\alpha_n)\in\mathcal{N}}$. It is not difficult to see that D_j is a nonempty closed and convex subset of $\widetilde{C}$ (notice that $\dot{x}_j \in D_j$). Futhermore, for a fixed $(\beta_n) \in \mathcal{N}$ and $\widetilde{y} \in D_j$,

$$\begin{aligned}
&\limsup_{(\alpha_n)\in\mathcal{N}} \|\widehat{T}_{(\alpha_n)}\dot{x}_j - \widehat{T}_{(\beta_n)}\widetilde{y}\| = \limsup_{(\alpha_n)\in\mathcal{N}} \|\widehat{T}_{(\alpha_n+\beta_n)}\dot{x}_j - \widehat{T}_{(\beta_n)}\widetilde{y}\| \\
&\leq \limsup_{(\alpha_n)\in\mathcal{N}} \|\widehat{T}_{(\alpha_n)}\dot{x}_j - \widetilde{y}\| \leq \varepsilon^j,
\end{aligned}$$

and hence $\widehat{T}_{(\beta_n)}(D_j) \subset D_j$ for every $(\beta_n) \in \mathcal{N}$. Again, by Bruck's theorem, there exists $\widetilde{y}_j \in D_j$ such that $\widehat{T}_{(\alpha_n)}\widetilde{y}_j = \widetilde{y}_j$ for all $(\alpha_n) \in \mathcal{N}$. Notice that $\|\widetilde{y}_j - \dot{x}_j\| \leq 2\varepsilon^j < 3\varepsilon^j$ and by Lemma 2.2, there exist $x_{j+1} \in C$ and $n_{j+1} \in \mathbb{N}$ such that $\|x_{j+1} - x_j\| < 3\varepsilon^j$ and $\|T^n x_{j+1} - x_{j+1}\| < \varepsilon^{j+1}$ for every $n \geq n_{j+1}$.

Thus we obtain by induction a sequence (n_j) of natural numbers and a sequence (x_j) of elements in C such that (4) is satisfied. It follows that (x_j) is a Cauchy sequence converging to some $x \in C$. Hence

$$\begin{aligned}
\|T^n x - x\| &\leq \|T^n x - T^n x_j\| + \|T^n x_j - x_j\| + \|x_j - x\| \\
&\leq (\|T^n x - T^n x_j\| - \|x_j - x\|) + \varepsilon^j + 2\|x_j - x\|
\end{aligned}$$

for every $n \geq n_j, j \geq 1$ and, consequently, $\lim_{n\to\infty} \|T^n x - x\| = 0$. Since T is continuous, $Tx = x$. □

A Banach space X is said to have the super fixed point property for asymptotically nonexpansive mappings (in the intermediate sense) if every Banach space Y which is finitely representable in X has the fixed point property for asymptotically nonexpansive mappings (in the intermediate sense). We can strengthen Theorem 2.3 in the following way.

Theorem 2.4. *A Banach space X has the super fixed point property for nonexpansive mappings if and only if X has the super fixed point property for asymptotically nonexpansive mappings in the intermediate sense.*

Proof. Assume that X has SFPP for nonexpansive mappings and let $\mathcal{U}$ be an ultrafilter defined on a set I. By Theorem 2.1, $(X)_{\mathcal{U}}$ has SFPP, too, and it follows from Theorem 2.3 that $(X)_{\mathcal{U}}$ has the fixed point property for asymptotically nonexpansive mappings in the intermediate sense. By Theorem 2.1 again, X has the super fixed point property for asymptotically nonexpansive mappings in the intermediate sense. The reverse implication is obvious. □

We conclude this section by giving some consequences of Theorem 2.3. Recall [14] that a Banach space is uniformly nonsquare if

$$\sup_{x,y\in S_X} \min\left\{\|x+y\|, \|x-y\|\right\} < 2.$$

García Falset, Lloréns Fuster and Mazcuñan Navarro (see [8]) solved a long-standing problem in metric fixed point theory by proving that uniformly nonsquare Banach spaces have FPP and, in consequence, SFPP for nonexpansive mappings.

Corollary 2.5. *Let C be a nonempty bounded closed and convex subset of a uniformly nonsquare Banach space. Then every asymptotically nonexpansive in the intermediate sense mapping $T : C \to C$ has a fixed point.*

In [19], Prus introduced the notion of uniformly noncreasy spaces. A real Banach space X is said to be uniformly noncreasy if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $f, g \in S_{X^*}$ and $\|f - g\| \geq \varepsilon$, then $\operatorname{diam} S(f, g, \delta) \leq \varepsilon$, where

$$S(f, g, \delta) = \{x \in B_X : f(x) \geq 1-\delta \ \wedge\ g(x) \geq 1-\delta\}$$

($\operatorname{diam}\emptyset = 0$). It is known that both uniformly convex and uniformly smooth spaces are uniformly noncreasy. The Bynum space $l^{2,\infty}$, which is l^2 space with the norm

$$\|x\|_{2,\infty} = \max\left\{\|x^+\|_2, \|x^-\|_2\right\},$$

and the space $X_{\sqrt{2}}$, which is l^2 space with the norm

$$\|x\|_{\sqrt{2}} = \max\left\{\|x\|_2, \sqrt{2}\|x\|_\infty\right\},$$

are examples of uniformly noncreasy spaces without normal structure. It was proved in [19] that all uniformly noncreasy spaces are superreflexive and have SFPP. This yields

Corollary 2.6. *Let C be a nonempty bounded closed and convex subset of a uniformly noncreasy Banach space. Then every asymptotically nonexpansive in the intermediate sense mapping $T : C \to C$ has a fixed point.*

Recently, a fixed point theorem in direct sums of two Banach spaces was proved in [23]. Assume that X has SFPP (for nonexpansive mappings) and Y is uniformly convex, uniformly smooth or finite dimensional. Since uniformly convex, uniformly smooth as well as finite dimensional spaces are

stable under passing to the Banach space ultrapowers and have uniform normal structure, it follows from [23, Th. 3.4], that $X \oplus Y$ with a strictly monotone norm has SFPP. Thus we obtain the following theorem.

Corollary 2.7. *Assume that X has SFPP and Y is uniformly convex, uniformly smooth or finite dimensional. Then $X \oplus Y$ with a strictly monotone norm has the fixed point property for asymptotically nonexpansive mappings in the intermediate sense.*

3. Common fixed points

In this section we generalize Theorem 2.3 for a commuting family of mappings. Let $\{T_t : t \in T\}$ be a commuting family of asymptotically nonexpansive self-mappings in the intermediate sense acting on a nonempty bounded closed and convex subset C of a Banach space X. Consider the set

$$\begin{aligned}\mathcal{A} = \{\{(t_1, \alpha_1), (t_2, \alpha_2), ..., (t_k, \alpha_k)\} : t_1, ..., t_k \in T,\, t_i \neq t_j \text{ for } i \neq j,\\ \alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{N}, k > 0\},\end{aligned}$$

directed by the relation

$$\{(t_1, \alpha_1), (t_2, \alpha_2), ..., (t_k, \alpha_k)\} \sqsubseteq \{(s_1, \beta_1), (s_2, \beta_2), ..., (s_m, \beta_m)\}$$

iff

$$\{t_1, t_2, ..., t_k\} \subseteq \{s_1, s_2, ... s_m\} \text{ and } \forall i\ \forall j\ (t_i = s_j \Rightarrow \alpha_i \leq \beta_j).$$

If $v = \{(t_1, \alpha_1), (t_2, \alpha_2), ..., (t_k, \alpha_k)\} \in \mathcal{A}$, write

$$T_v x = T_{t_1}^{\alpha_1} T_{t_2}^{\alpha_2} ... T_{t_k}^{\alpha_k} x,$$

and let

$$\mathcal{D} = \left\{(v_n) \in \mathcal{A}^{\mathbb{N}} : \limsup_{n\to\infty} \sup_{x,y\in C} (\|T_{v_n}x - T_{v_n}y\| - \|x-y\|) \leq 0\right\}.$$

Note that $\mathcal{D} \neq \emptyset$ since $(\{(t, n)\})_{n\in\mathbb{N}} \in \mathcal{D}$ for every $t \in T$. If $(v_n), (u_n) \in \mathcal{D}$, define the relation $(v_n) \preceq (u_n)$ iff $v_n \sqsubseteq u_n$ for every $n \in \mathbb{N}$. It is not difficult to see that for every $(v_n), (u_n) \in \mathcal{D}$ there exists $(w_n) \in \mathcal{D}$ such that $(v_n) \preceq (w_n)$ and $(u_n) \preceq (w_n)$. Indeed, let

$$\begin{aligned}v_n &= \{(t_1^{(n)}, \alpha_1^{(n)}), (t_2^{(n)}, \alpha_2^{(n)}), ..., (t_{k_n}^{(n)}, \alpha_{k_n}^{(n)})\},\\ u_n &= \{(s_1^{(n)}, \beta_1^{(n)}), (s_2^{(n)}, \beta_2^{(n)}), ..., (s_{m_n}^{(n)}, \beta_{m_n}^{(n)})\},\end{aligned}$$

and put

$$w_n = \{(t_1^{(n)}, \alpha_1^{(n)}), (t_2^{(n)}, \alpha_2^{(n)}), ..., (t_{k_n}^{(n)}, \alpha_{k_n}^{(n)}), (s_1^{(n)}, \beta_1^{(n)}), (s_2^{(n)}, \beta_2^{(n)}), ..., (s_{m_n}^{(n)}, \beta_{m_n}^{(n)})\}, \tag{5}$$

$n \in \mathbb{N}$ (to shorten notation, we use the convention that if $t_i = s_j$ for some i, j, then the pairs $(t_i, \alpha_i), (s_j, \beta_j)$ in w_n are identified with one pair $(t_i, \alpha_i + \beta_j)$). Notice that

$$\begin{aligned}s((T_{w_n})) &= \limsup_{n\to\infty} \sup_{x,y\in C} (\|T_{v_n}T_{u_n}x - T_{v_n}T_{u_n}y\| - \|x-y\|)\\ &\leq s((T_{v_n})) + s((T_{u_n})) \leq 0,\end{aligned}$$

where

$$s((T_{v_n})) = \limsup_{n\to\infty} \sup_{x,y\in C} (\|T_{v_n}x - T_{v_n}y\| - \|x-y\|).$$

Hence $(w_n) \in \mathcal{D}$ and, clearly, $(v_n) \preceq (w_n)$ and $(u_n) \preceq (w_n)$. Thus $(\mathcal{D}, \preceq)$ is a directed set.

Let p be a free ultrafilter on $\mathbb{N}$. Then, for every $(x_n)_p, (y_n)_p \in \widetilde{C}$ and $(v_n) \in \mathcal{D}$,

$$\lim_{n\to p}(\|T_{v_n}x_n - T_{v_n}y_n\| - \|x_n - y_n\|) \le \limsup_{n\to\infty} \sup_{x,y\in C} (\|T_{v_n}x - T_{v_n}y\| - \|x-y\|) \le 0. \tag{6}$$

Therefore, we may define unambiguously a mapping $\widehat{T}_{(v_n)} : \widetilde{C} \to \widetilde{C}$, by setting

$$\widehat{T}_{(v_n)}(x_n)_p = (T_{v_n}x_n)_p. \tag{7}$$

It follows from (6) that $\widehat{T}_{(v_n)}$ is nonexpansive for every $(v_n) \in \mathcal{D}$.

We can now prove a counterpart of Lemma 2.2.

Lemma 3.1. *Let $\{T_t : t \in T\}$ be a commuting family of asymptotically nonexpansive mappings in the intermediate sense acting on a nonempty bounded closed and convex subset C of a Banach space X. Suppose that there exists $\widetilde{y} \in \widetilde{C}$ such that*

$$\widehat{T}_{(v_n)}\widetilde{y} = \widetilde{y} \tag{8}$$

for all $(v_n) \in \mathcal{D}$. Let $\|\widetilde{y} - \dot{x}_0\| < \delta$ for some $x_0 \in C$ and $\delta > 0$. Then, for every $\varepsilon > 0$ there exist $x \in C$ and $n \in \mathbb{N}$ such that $\|x - x_0\| < \delta$ and $\|T_u x - x\| < \varepsilon$ for every $u \in \mathcal{D}'(n)$, where

$$\mathcal{D}'(n) = \{v = \{(t_1,\alpha_1),(t_2,\alpha_2),...,(t_k,\alpha_k)\} \in \mathcal{A} :$$
$$\sup_{x,y\in C}(\|T_v x - T_v y\| - \|x-y\|) \le \frac{1}{n+1}\}.$$

Proof. Since $\|\widetilde{y} - \dot{x}_0\| < \delta$, there exists a sequence (y_n) in C such that $\|y_n - x_0\| < \delta$ for all $n \in \mathbb{N}$ and $\widetilde{y} = (y_n)_p$. Assume, conversely to our claim, that there exists $\varepsilon_0 > 0$ such that for every $x \in C$ and $n \in \mathbb{N}$ there exists $u \in \mathcal{D}'(n)$ such that $\|x - x_0\| \ge \delta$ or $\|T_u x - x\| \ge \varepsilon_0$. We shall define a sequence $(u_n) \in \mathcal{D}$ by induction. For $n = 0$ and $y_0 \in C$, there exists $u_0 \in \mathcal{D}'(0)$ such that $\|T_{u_0}y_0 - y_0\| \ge \varepsilon_0$. Suppose that we have chosen $u_0 \in \mathcal{D}'(0), u_1 \in \mathcal{D}'(1), ..., u_n \in \mathcal{D}'(n)$ such that $\|T_{u_i}y_i - y_i\| \ge \varepsilon_0$ for $i = 0,1,...,n$. By assumption, since $\|y_{n+1} - x_0\| < \delta$, there exists $u_{n+1} \in \mathcal{D}'(n+1)$ such that $\|T_{u_{n+1}}y_{n+1} - y_{n+1}\| \ge \varepsilon_0$. Thus we obtain a sequence $(u_n) \in \mathcal{A}^{\mathbb{N}}$ such that $\|T_{u_n}y_n - y_n\| \ge \varepsilon_0$ and

$$\sup_{x,y\in C}(\|T_{u_n}x - T_{u_n}y\| - \|x-y\|) \le \frac{1}{n+1}$$

for all $n \in \mathbb{N}$. Hence

$$\limsup_{n\to\infty} \sup_{x,y\in C}(\|T_{u_n}x - T_{u_n}y\| - \|x-y\|) = 0,$$

i. e., $(u_n) \in \mathcal{D}$. But this contradicts (8), since $\left\|\widehat{T}_{(u_n)}\widetilde{y} - \widetilde{y}\right\| \ge \varepsilon_0$. □

We will also make use of the following simple observation.

Lemma 3.2. *For every* $(u_n) \in \mathcal{D}$ *and* $\tilde{x}, \tilde{y} \in \widetilde{C}$,

$$\limsup_{(v_n)\in\mathcal{D}} \|\widehat{T}_{(v_n)}\tilde{x} - \widehat{T}_{(u_n)}\tilde{y}\| = \limsup_{(v_n)\in\mathcal{D}} \|\widehat{T}_{(v_n)}\widehat{T}_{(u_n)}\tilde{x} - \widehat{T}_{(u_n)}\tilde{y}\|.$$

Proof. Fix $(u_n) \in \mathcal{D}$, $\tilde{x}, \tilde{y} \in \widetilde{C}$, and notice that for every $(v_n) \in \mathcal{D}$,

$$\sup_{(\bar{w}_n)\succeq(w_n)} \|\widehat{T}_{(\bar{w}_n)}\tilde{x} - \widehat{T}_{(u_n)}\tilde{y}\| = \sup_{(\bar{v}_n)\succeq(v_n)} \|\widehat{T}_{(\bar{v}_n)}\widehat{T}_{(u_n)}\tilde{x} - \widehat{T}_{(u_n)}\tilde{y}\|,$$

where (w_n) is defined by (5). Hence

$$\limsup_{(v_n)\in\mathcal{D}} \|\widehat{T}_{(v_n)}\tilde{x} - \widehat{T}_{(u_n)}\tilde{y}\| \le \limsup_{(v_n)\in\mathcal{D}} \|\widehat{T}_{(v_n)}\widehat{T}_{(u_n)}\tilde{x} - \widehat{T}_{(u_n)}\tilde{y}\|.$$

The reverse inequality is obvious since $\widehat{T}_{(v_n)}\widehat{T}_{(u_n)}\tilde{x} = \widehat{T}_{(w_n)}\tilde{x}$ and $(v_n) \preceq (w_n)$. ∎

We are now in a position to prove the following generalization of Theorem 2.3.

Theorem 3.3. *Suppose C is a nonempty bounded closed and convex subset of a Banach space X with SFPP and $\mathcal{T} = \{T_t : t \in T\}$ is a commuting family of asymptotically nonexpansive mappings in the intermediate sense acting on C. Then there exists $x \in C$ such that $T_t x = x$ for every $t \in T$ (a common fixed point for $\mathcal{T}$).*

Proof. We partly follow the reasoning given in the proof of Theorem 2.3. Assume that X has the super fixed point property for nonexpansive mappings. Let $\mathcal{T} = \{T_t : t \in T\}$ be a commuting family of asymptotically nonexpansive mappings in the intermediate sense acting on a nonempty bounded closed and convex set $C \subset X$. We can assume that $\operatorname{diam} C = 1$. Take a free ultrafilter p on $\mathbb{N}$, $(v_n) \in \mathcal{D}$ and define $\widehat{T}_{(v_n)}$ by (7). Notice that for every $(v_n), (u_n) \in \mathcal{D}$ and any $(x_n)_p \in \widetilde{C}$,

$$(\widehat{T}_{(v_n)} \circ \widehat{T}_{(u_n)})(x_n)_p = (\widehat{T}_{(u_n)} \circ \widehat{T}_{(v_n)})(x_n)_p.$$

It follows from Bruck's theorem that there exists $\widetilde{y}_0 \in \widetilde{C}$ such that $\widehat{T}_{(v_n)}\widetilde{y}_0 = \widetilde{y}_0$ for all $(v_n) \in \mathcal{D}$. Fix $\varepsilon < 1$ and $x_0 \in C$. We shall define by induction a sequence (n_j) of natural numbers and a sequence (x_j) of elements in C such that

$$\|x_j - x_{j-1}\| < 3\varepsilon^{j-1} \text{ and } \|T_u x_j - x_j\| < \varepsilon^j \text{ for every } u \in \mathcal{D}'(n_j), j \ge 1. \tag{9}$$

By Lemma 3.1, there exist $x_1 \in C$ and $n_1 \in \mathbb{N}$ such that $\|T_u x_1 - x_1\| < \varepsilon$ for every $u \in \mathcal{D}'(n_1)$ and $\|x_1 - x_0\| \le \operatorname{diam} C < 3$.

Suppose that we have chosen natural numbers $n_1, ..., n_j$ and $x_1, ..., x_j \in C$ $(j \ge 1)$ such that

$$\|x_i - x_{i-1}\| < 3\varepsilon^{i-1} \text{ and } \|T_u x_i - x_i\| < \varepsilon^i \text{ for every } u \in \mathcal{D}'(n_i), 1 \le i \le j. \tag{10}$$

Let

$$D_j = \left\{ \widetilde{y} = (y_n)_p \in \widetilde{C} : \limsup_{(v_n)\in\mathcal{D}} \|\widehat{T}_{(v_n)}\dot{x}_j - \widetilde{y}\| \le \varepsilon^j \right\},$$

where $\dot{x}_j = (x_j, x_j, ...)_p$ and

$$\limsup_{(v_n)\in\mathcal{D}} \|\widehat{T}_{(v_n)}\dot{x}_j - \widetilde{y}\| = \inf_{(v_n)\in\mathcal{D}} \sup_{(u_n)\succeq(v_n)} \lim_{n\to p} \|T_{u_n}x_j - y_n\| .$$

Notice that for every $(v_n) \in \mathcal{D}$ and $\eta > 0$, there exists $k \in \mathbb{N}$ such that $\sup_{x,y\in C}(\|T_{v_n}x - T_{v_n}y\| - \|x-y\|) < \eta$ for every $n > k$. Hence $v_n \in \mathcal{D}'(n_j)$ for sufficiently large n and applying the induction assumption (10) gives $\lim_{n\to p} \|T_{v_n}x_j - x_j\| \le \varepsilon^j$ for every $(v_n) \in \mathcal{D}$. It follows that $\dot{x}_j \in D_j$ and D_j is a nonempty closed and convex subset of $\widetilde{C}$. By Lemma 3.2, for fixed $(u_n) \in \mathcal{D}$ and $\widetilde{y} \in D_j$,

$$\begin{aligned}
&\limsup_{(v_n)\in\mathcal{D}} \|\widehat{T}_{(v_n)}\dot{x}_j - \widehat{T}_{(u_n)}\widetilde{y}\| = \limsup_{(v_n)\in\mathcal{D}} \|\widehat{T}_{(v_n)}\widehat{T}_{(u_n)}\dot{x}_j - \widehat{T}_{(u_n)}\widetilde{y}\| \\
&\le \limsup_{(v_n)\in\mathcal{D}} \|\widehat{T}_{(v_n)}\dot{x}_j - \widetilde{y}\| \le \varepsilon^j,
\end{aligned}$$

and hence $\widehat{T}_{(u_n)}(D_j) \subset D_j$ for every $(u_n) \in \mathcal{D}$. By Bruck's theorem, there exists $\widetilde{y}_j \in D_j$ such that $\widehat{T}_{(v_n)}\widetilde{y}_j = \widetilde{y}_j$ for all $(v_n) \in \mathcal{D}$. It is easy to see that $\|\widetilde{y}_j - \dot{x}_j\| < 3\varepsilon^j$ and, by Lemma 3.1, there exist $x_{j+1} \in C$ and $n_{j+1} \in \mathbb{N}$ such that $\|x_{j+1} - x_j\| < 3\varepsilon^j$ and $\|T_u x_{j+1} - x_{j+1}\| < \varepsilon^{j+1}$ for every $u \in \mathcal{D}'(n_{j+1})$. Thus we obtain by induction a sequence (n_j) of natural numbers and a sequence (x_j) of elements in C such that (9) is satisfied. It follows that (x_j) is a Cauchy sequence converging to some $x \in C$.

Fix $T_t \in \mathcal{T}$ and notice that for every n_j, there exists k_j such that $\{(t,n)\} \in \mathcal{D}'(n_j)$ for $n > k_j$, since T_t is asymptotically nonexpansive in the intermediate sense. Applying (9) gives

$$\limsup_{n\to\infty} \|T_t^n x_j - x_j\| \le \varepsilon^j,\ j \ge 1.$$

Furthermore,

$$\begin{aligned}
\|T_t^n x - x\| &\le (\|T_t^n x - T_t^n x_j\| - \|x - x_j\|) + \|T_t^n x_j - x_j\| + 2\|x_j - x\| \\
&\le \sup_{x,y\in C} (\|T_t^n x - T_t^n y\| - \|x-y\|) + \|T_t^n x_j - x_j\| + 2\|x_j - x\|.
\end{aligned}$$

for every $j, n \ge 1$ and, consequently, $\limsup_{n\to\infty} \|T_t^n x - x\| = 0$. Since T_t is continuous, $T_t x = x$. □

Remark. It was proved in [5, Th. 4] that if C is a nonempty weakly compact convex subset of a Banach space X and every asymptotically nonexpansive mapping of C satisfies the (ω)-fixed point property (which is a little stronger than the fixed point property), then the set of common fixed points of any commuting family of asymptotically nonexpansive mappings acting on C is a nonexpansive retract of C. It is not known whether a similar conclusion can be drawn under the assumptions of Theorem 3.3.

References

[1] A. G. Aksoy, M. A. Khamsi, Nonstandard Methods in Fixed Point Theory, Springer-Verlag, New York-Berlin, 1990.

[2] J. M. Ayerbe Toledano, T. Domínguez Benavides, G. López Acedo, Measures of Noncompactness in Metric Fixed Point Theory, Birkhäuser Verlag, Basel, 1997.

[3] R. E. Bruck, Jr., A common fixed point theorem for a commuting family of nonexpansive mappings, Pacific J. Math. 53 (1974), 59–71.

[4] R. Bruck, T. Kuczumow, S. Reich, Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property, Colloq. Math. 65 (1993), 169–179.

[5] T. Domínguez Benavides, P. Lorenzo Ramírez, Structure of the fixed point set and common fixed points of asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 129 (2001), 3549–3557

[6] P. N. Dowling, C. J. Lennard, B. Turett, Some fixed point results in ℓ_1 and c_0, Nonlinear Anal. 39 (2000), 929–936.

[7] D. van Dulst and A. J. Pach, On flatness and some ergodic super-properties of Banach spaces, Indag. Math. 43 (1981), 153-164.

[8] J. García Falset, E. Lloréns Fuster, E. M. Mazcuñan Navarro, Uniformly nonsquare Banach spaces have the fixed point property for nonexpansive mappings, J. Funct. Anal. 233 (2006), 494–514.

[9] K. Goebel, W. A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 35 (1972), 171–174.

[10] K. Goebel, W. A. Kirk, Topics in Metric Fixed Point Theory, Cambridge University Press, Cambridge, 1990.

[11] Handbook of Metric Fixed Point Theory, W. A. Kirk, B. Sims (eds.), Kluwer Academic Publishers, Dordrecht, 2001.

[12] S. Heinrich, Ultraproducts in Banach space theory, J. Reine and Ang. Mat. 313 (1980), 72-104.

[13] C. W. Henson and L. C. Moore, Jr., Subspaces of the nonstandard hull of a normed space, Trans. Amer. Math. Soc. 197 (1974), 131-143.

[14] R. C. James, Uniformly non-square Banach spaces, Ann. of Math. 80 (1964), 542–550.

[15] T. H. Kim, H. K. Xu, Remarks on asymptotically nonexpansive mappings, Nonlinear Anal. 41 (2000), 405–415.

[16] W. A. Kirk, C. Martinez Yañez, S. S. Shin, Asymptotically nonexpansive mappings, Nonlinear Anal. 33 (1998), 1–12.

[17] P. K. Lin, K. K. Tan, H. K. Xu, Demiclosedness principle and asymptotic behavior for asymptotically nonexpansive mappings. Nonlinear Anal. 24 (1995), 929–946.

[18] B. Maurey, Points fixes des contractions de certains faiblement compacts de L^1, Semin. Anal. Fonct. 1980-1981, Ecole Polytechnique, Palaiseau, 1981.

[19] S. Prus, Banach spaces which are uniformly noncreasy, in: Proc. 2nd World Congress of Nonlinear Analysts, V. Lakshmikantham (ed.), Nonlinear Anal. **30** (1997), 2317–2324.

[20] B. Sims, Ultra-techniques in Banach Space Theory, Queen's University, Kingston, Ontario, 1982.

[21] J. Stern, Some applications of model theory in Banach space theory, Ann. Math. Logic 9 (1976), 49-122.

[22] H. K. Xu, Existence and convergence for fixed points of mappings of asymptotically nonexpansive type, Nonlinear Anal. 16 (1991), 1139–1146.

[23] A. Wiśnicki, On the fixed points of nonexpansive mappings in direct sums of Banach spaces, Studia Math. 207 (2011), 75-84.

Andrzej Wiśnicki, Institute of Mathematics, Maria Curie-Skłodowska University, 20-031 Lublin, Poland
E-mail address: awisnic@hektor.umcs.lublin.pl